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Zak Transforms on Binary Fields

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We introduce analogues of the Zak Transform on binary fields, and show that they are bounded linear operators on L^p for $p=1$ and 2. We also show that positivity of Zak transforms can be used to decide whether orthonormal systems generated by multiplying characters of **F** by a weight function are complete or not. © 1999 Academic Press

1. INTRODUCTION

Let **C** denote the complex plane, $\mathbf{Z} := \{0, \pm 1, \pm 2, ...\}$ denote the set of integers, and $\mathbf{N} := \{0, 1, 2, ...\}$ denote the set of natural numbers. By a *binary field* **F** we mean either the 2-series field $(\mathbf{F}, \mathring{+}, \circ)$ or the 2-adic field $(\mathbf{F}, \dot{+}, \bullet)$, i.e., the set of doubly infinite sequences

$$\mathbf{F} := \{(a^{(j)}, j \in \mathbf{Z}) : a^{(j)} = 0 \text{ or } 1 \text{ and } \lim_{j \to -\infty} a^{(j)} = 0\},$$

whose field structures are generated by identifying an element $(a^{(k)}, k \in \mathbf{Z})$ in **F** with a Laurent series

$$\sum_{k=-\infty}^{\infty} a^{(k)} 2^k$$

and formally adding and multiplying these Laurent series with one difference: in the 2-series case no carrying takes place but in the 2-adic case, one carries from left to right. These fields are non-Archimedean normed local fields which serve as prototypes for all nonclassical nondiscrete locally

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compact fields with reasonable topology. The norm of an element $(a^{(j)}, j \in \mathbf{Z})$ is 2^{-l}, where l is the integer which satisfies $a^{(l)} = 1$ and $a^{(j)} = 0$ for all $j < l$ (see Taibleson [8] for details).

The Zak transform was introduced on the field $\mathbf{R} := (-\infty, \infty)$ by Zak [11] in connection with solid state physics. Since that time it has been used widely (both explicitly and implicitly) in numerous mathematical and applied articles. It has also been called the Weil-Brezin map and some have traced its history back to Gauss. (For more details and the history of this subject, see Janssen [6].)

We shall introduce two types of Zak transforms on binary fields. The first type is a direct adaptation of the Zak transform from the field $(\mathbf{R}, +, \cdot)$ to the 2-series field $(\mathbf{F}, \overset{\circ}{+}, \circ)$. The second type, which will be introduced on both the 2-adic field $(\mathbf{F}, \overset{\bullet}{+}, \bullet)$ and the 2-series field $(\mathbf{F}, \overset{\circ}{+}, \circ)$, is similar except the roles of the integers and the unit interval are reversed. For the only common case, the 2-series field, we shall see that the second type of Zak transform is in some sense a dual of the first.

This is the first time the Zak transform has been examined in this generality. Tolimieri and An [9] have studied an analogue of the second type (or "dual") Zak transform on finite abelian groups, obtaining a discrete version of Theorem 5(ii) below. By dealing with finite groups only, they avoid all questions of convergence. In fact, their aims are fundamentally different than ours. They use the Zak transform to characterize Weyl–Heisenberg systems and apply it to design of algorithms and codes. We, of course, are interested in the connection between Zak transforms and the harmonic analysis of binary fields, and the role it plays in dyadic wavelets.

2. NOTATION AND FOLKLORE

In this section we introduce enough notation and background to describe our results. To unify our presentation, we shall use the generic notation $(\mathbf{F}, +, \cdot)$ to denote both binary fields and the classical field $\mathbf{R}$. Details for binary fields can be found in [8], [5], and [10]. Details for the classical real field can be found in [12].

Let $\mathbf{I}$ represent the unit interval of $\mathbf{F}$, i.e., $\mathbf{I} := [0, 1)$ when $\mathbf{F} = \mathbf{R}$ and $\mathbf{I}$ is the set of $(a^{(j)}, j \in \mathbf{Z})$ such that $a^{(j)} = 0$ for $j \geqslant 0$ when $\mathbf{F}$ is one of the binary fields. Let $\mathbf{F}^{\#}$ denote the set of integers of $\mathbf{F}$, i.e., $\mathbf{F}^{\#} := \mathbf{Z}$ when $\mathbf{F} = \mathbf{R}$ and $\mathbf{F}^{\#}$ is the set of $(a^{(j)}, j \in \mathbf{Z})$ such that $a^{(j)} = 0$ for $j < 0$ when $\mathbf{F}$ is one of the binary fields. It is well known that the dual group of the additive group $(\mathbf{F}, +)$ is $\mathbf{F}$ itself, and that the dual group of the additive group $(\mathbf{I}, +)$ is $\mathbf{F}^{\#}$ (one must use modulo arithmetic on $\mathbf{I}$ when $\mathbf{F} = \mathbf{R}$). In Section 4 we shall use the notation $\mathcal{N} := \mathbf{F}^{\#} \times \mathbf{F}^{\#}$.

Each of these fields has a *basic character*, i.e., a single function $u\colon \mathbf{F} \to \mathbf{C}$ such that the set of characters of the additive group $(\mathbf{F}, +)$ are given by $u_y(x) := u(x \cdot y)$ as y ranges over $\mathbf{F}$ and the set of characters of the additive group $(\mathbf{I}, +)$ are given by $u_k(x) := u(x \cdot k)$ as k ranges over $\mathbf{F}^\#$. Indeed, for the case $\mathbf{F} = \mathbf{R}$, $u(x) := \exp(2\pi i x)$. When $\mathbf{F}$ is the 2-series field, $u(x) := \exp(2\pi i((x^{(-1)}/2) + (x^{(-2)}/2^2) + \cdots))$. And when $\mathbf{F}$ is the 2-series field, $u(x) := \exp(2\pi i x^{(-1)})$. (See [4], [8] or [10] for details.)

Notice that the map $(t, k) \mapsto t + k$ is a 1–1 function from $\mathbf{F}^\# \times \mathbf{I}$ onto $\mathbf{F}$, hence we can identify $\mathbf{F}$ with either $\mathbf{I} \times \mathbf{F}^\#$ or $\mathbf{F}^\# \times \mathbf{I}$. Under these identifications, a function $g\colon \mathbf{F} \to \mathbf{C}$ is taken to functions defined on $\mathbf{I} \times \mathbf{F}^\#$ or on $\mathbf{F}^\# \times \mathbf{I}$ by $(\tau_k g)(t) := g(t + k)$, where $t \in \mathbf{I}$ and $k \in \mathbf{F}^\#$. These functions can in turn be identified with functions on $\mathbf{F} \times \mathbf{F}^\#$ or on $\mathbf{F}^\# \times \mathbf{F}$ which are given by $\chi \tau_k g$, where $\chi := \chi_{\mathbf{I}}$.

We can use this point of view to introduce mixed norm spaces in the following way. For each pair $p, q \in \{1, \infty\}$ with $p \neq q$, define four different norms for measurable functions defined on $\mathbf{F}$:

$$\|g\|_{1\infty} := \sup_{k \in \mathbf{F}^\#} \|\chi \tau_k g\|_1, \qquad \|g\|_{\infty 1} := \sum_{k \in \mathbf{F}^\#} \|\chi \tau_k g\|_\infty,$$

$$\|g\|_{\langle 1\infty \rangle} := \Big\| \sum_{k \in \mathbf{F}^\#} |\chi \tau_k g| \Big\|_\infty, \qquad \|g\|_{\langle \infty 1 \rangle} := \Big\| \sup_{k \in \mathbf{F}^\#} |\chi \tau_k g| \Big\|_1,$$

where $\|\cdot\|_p$ denotes the $L^p(\mathbf{F})$ norm. Let $L^{pq}(\mathbf{F})$, respectively $L^{\langle pq \rangle}(\mathbf{F})$, denote the collection of g such that $\|g\|_{pq} < \infty$, respectively, $\|g\|_{\langle pq \rangle} < \infty$. Clearly,

$$\|g\|_{1\infty} \leqslant \|g\|_{\langle \infty 1 \rangle} \leqslant \|g\|_{11} = \|g\|_1 \leqslant \|g\|_{\langle 1\infty \rangle} \leqslant \|g\|_{\infty 1}, \tag{1}$$

i.e., $L^{\infty 1}(\mathbf{F}) \subset L^{\langle 1\infty \rangle}(\mathbf{F}) \subset L^1(\mathbf{F}) \subset L^{\langle \infty 1 \rangle}(\mathbf{F}) \subset L^{1\infty}(\mathbf{F})$. When $\mathbf{F} = \mathbf{R}$, the spaces $L^{pq}(\mathbf{R})$ are called *amalgam spaces* or *Wiener-type spaces*, and are sometimes denoted by $W(L^p, L^q)$. They were used in the special cases $p = \infty$, $q = 1$ by Wiener (see Feichtinger [3]).

3. THE ZAK TRANSFORM

To define the Zak transform on $\mathbf{R}$ and the 2-series field, let $(\mathbf{F}, +, \cdot)$ represent either $(\mathbf{R}, +, \cdot)$ or $(\mathbf{F}, \mathring{+}, \circ)$, λ denote Haar measure on $\mathbf{F}$, $I_k(t)$ represent the interval in $\mathbf{F}$ of Haar measure 2^{-k} which contains $t \in \mathbf{F}$ (for binary fields, $I_k(t)$ is the set of $(a^{(j)}, j \in \mathbf{Z})$ such that $a^{(j)} = t^{(j)}$ for $j < k$), u

represent a basic character of $\mathbf{F}$, and $u_k(x) = u(k \cdot x)$ for $k \in \mathbf{F}^\#$, $x \in \mathbf{F}$. Then the *Zak transform* of an $f \in L^1(\mathbf{F})$ is defined by

$$(\mathscr{Z}f)(t, s) := \sum_{k \in \mathbf{F}^\#} f(t+k)\, u_k(s) \qquad (t, s \in \mathbf{F}). \tag{2}$$

Since

$$\sum_{k \in \mathbf{F}^\#} \int_{I_0(m)} |f(t+k)|\, d\lambda(t) = \|f\|_1 < \infty$$

for any $m \in \mathbf{F}$, it is clear that the series (2) converges absolutely for each $s \in \mathbf{F}$ and a.e. $[\lambda]$ $t \in \mathbf{F}$. Moreover, it is easy to check that $\mathscr{Z}$ is a bounded linear operator from $L^1(\mathbf{F})$ into the mixed norm space $L^{1\infty}(\mathbf{F}^2)$ with

$$\|\mathscr{Z}f\|_{1\infty} \leqslant \|f\|_1. \tag{3}$$

The values of $\mathscr{Z}$ are uniquely determined by its values on $\mathbf{I} \times \mathbf{I}$. First, since the characters of the real and 2-series fields satisfy $u_k(s+m) = u_k(s)$ for all $s \in \mathbf{F}$ and $k, m \in \mathbf{F}^\#$, it is clear that

$$(\mathscr{Z}f)(t, s+m) = (\mathscr{Z}f)(t, s) \qquad (s, t \in \mathbf{F}, m \in \mathbf{F}^\#).$$

Next, since $\mathbf{F}^\#$ is a subgroup of $(\mathbf{F}, +)$, we have for each $s, t \in \mathbf{F}$ and $n \in \mathbf{F}^\#$ that

$$\begin{aligned}(\mathscr{Z}f)(t+n, s) &= \sum_{k \in \mathbf{F}^\#} f(t+n+k)\, u_k(s)\\ &= \bar{u}_n(s) \sum_{k \in \mathbf{F}^\#} f(t+n+k)\, u_{n+k}(s)\\ &= \bar{u}_n(s) \sum_{l \in \mathbf{F}^\#} f(t+l)\, u_l(s) = \bar{u}_n(s)(\mathscr{Z}f)(t, s).\end{aligned}$$

Therefore,

$$(\mathscr{Z}f)(t, s+m) = (\mathscr{Z}f)(t, s), \qquad (\mathscr{Z}f)(t+n, s) = \bar{u}_n(s)(\mathscr{Z}f)(t, s) \tag{4}$$

for all $t, s \in \mathbf{F}$ and $n, m \in \mathbf{F}^\#$. In particular, for each $f \in L^1(\mathbf{F})$ the values of $\mathscr{Z}f$ are completely determined by its values on $\mathbf{I}^2 := \mathbf{I} \times \mathbf{I}$.

The transform $\mathscr{Z}$ can be extended in a natural way to $L^1(\mathbf{F}) \cup L^2(\mathbf{F})$. In fact, we notice that if $f \in L^2(\mathbf{F})$, then

$$\sum_{k \in \mathbf{F}^\#} \int_{\mathbf{I}} |f(t+k)|^2\, d\lambda(t) = \|f\|_2^2 < \infty.$$

Hence

$$\sum_{k \in \mathbf{F}^{\#}} |f(t+k)|^2 < \infty$$

for a.e. $[\lambda]$ $t \in \mathbf{I}$. Since $(u_n, n \in \mathbf{F}^{\#})$ is a complete orthonormal system on $\mathbf{I}$, it follows that series (2) converges in $L^2(\mathbf{I})$ norm for each such t. Since every L^2 convergent sequence contains an a.e. convergent subsequence with the same limit, we see that the $L^2(\mathbf{I})$ limit of (2) and the a.e. $[\lambda]$ limit of (2) agree when $f \in L^1(\mathbf{F}) \cap L^2(\mathbf{I})$. In particular, we can also denote the $L^2(\mathbf{I})$ limit of (2) by $\mathscr{Z}f$. Now define $\mathscr{Z}$ on all of $L^1(\mathbf{F}) \cup L^2(\mathbf{F})$ by extending $\mathscr{Z}f$ to $\mathbf{F}$ using (4).

The following result contains a list of the Zak transform's most important properties.

THEOREM 1. *Let $\mathscr{Z}$ represent the Zak transform on the real or* 2*-series field. Then*

(i) *$\mathscr{Z}$ is a* 1–1 *bounded linear operator from $L^1(\mathbf{F})$ into $L^{1\infty}(\mathbf{I}^2)$ with*

$$\|\mathscr{Z}f\|_{1\infty} \leqslant \|f\|_1, \tag{5}$$

and

(ii) *$\mathscr{Z}$ is a unitary operator from $L^2(\mathbf{F})$ onto $L^2(\mathbf{I}^2)$, i.e.,*

$$\langle \mathscr{Z}f, \mathscr{Z}g \rangle = \langle f, g \rangle \qquad (f, g \in L^2(\mathbf{F})), \tag{6}$$

where the symbol on the left side of (6) *represents the inner product of the Hilbert space $L^2(\mathbf{I}^2)$.*

Proof. Except that $\mathscr{Z}$ is 1–1, part (i) has been verified. To show $\mathscr{Z}$ is 1–1, let $\mathscr{Z}f = 0$ for some $f \in L^1(\mathbf{F})$. Since (2) converges uniformly in s, for a.e. $[\lambda]$ $t \in \mathbf{I}$, and $(u_k, k \in \mathbf{F}^{\#})$ is orthonormal on $\mathbf{I}$, we have

$$0 = \langle (\mathscr{Z}f)(t, \cdot), u_k \rangle = f(t+k)$$

for all $k \in \mathbf{F}^{\#}$ and a.e. $[\lambda]$ $t \in \mathbf{I}$. Therefore, $f = 0$ a.e. $[\lambda]$ on $\mathbf{F}$.

To prove (ii), observe first that by Parseval's formula,

$$\|(\mathscr{Z}f)(t, \cdot)\|_2^2 = \sum_{k \in \mathbf{F}^{\#}} |f(t+k)|^2$$

for a.e. $[\lambda]$ $t \in \mathbf{I}$. Consequently,

$$\|\mathscr{Z}f\|_2 = \|f\|_2$$

for all $f \in L^2(\mathbf{F})$, i.e., $\mathscr{Z}$ is unitary. Next, notice since $(u_k, k \in \mathbf{F}^\sharp)$ is orthonormal on $\mathbf{I}$ that

$$\int_{\mathbf{I}} (\mathscr{Z}f)(t, s)(\overline{\mathscr{Z}}g(t, s))\, d\lambda(s) = \sum_{k \in \mathbf{F}^\sharp} f(t+k)\, \bar{g}(t+k)$$

for a.e. $[\lambda]$ $t \in \mathbf{I}$. Integrating $d\lambda(t)$ over $\mathbf{I}$, we obtain

$$\langle \mathscr{Z}f, \mathscr{Z}g \rangle = \sum_{k \in \mathbf{F}^\sharp} \int_{\mathbf{I}} f(t+k)\, \bar{g}(t+k)\, d\lambda(t).$$

Thus (6) holds.

To prove that $\mathscr{Z}$ is onto, let $F \in L^2(\mathbf{I}^2)$ and let $\tilde{I}$ denote the collection of all $t \in \mathbf{I}$ such that $F(t, \cdot) \in L^2(\mathbf{I})$. By hypothesis, $\lambda(\tilde{I}) = 1$ and for each $t \in \tilde{I}$ the one-variable function $F(t, \cdot)$ can be represented in $L^2(\mathbf{I})$ norm by the series

$$F(t, \cdot) = \sum_{k \in \mathbf{F}^\sharp} c_k(t)\, u_k,$$

where

$$c_k(t) := \int_{\mathbf{I}} F(t, s)\, \bar{u}_k(s)\, d\lambda(s) \qquad (k \in \mathbf{F}^\sharp).$$

Define a function $f\colon \mathbf{F} \to \mathbf{C}$ by

$$f(t+k) := c_k(t) \qquad (t \in \tilde{I}, k \in \mathbf{F}^\sharp)$$

and $f = 0$ elsewhere. By Parseval's formula,

$$\begin{aligned} \int_{\mathbf{F}} |f|^2\, d\lambda &= \sum_{k \in \mathbf{F}^\sharp} \int_{\mathbf{I}} |f(t+k)|^2\, d\lambda(t) \\ &= \sum_{k \in \mathbf{F}^\sharp} \int_{\mathbf{I}} |c_k(t)|^2\, d\lambda(t) \\ &= \int_{\mathbf{I}} \int_{\mathbf{I}} |F(t, s)|^2\, d\lambda(s)\, d\lambda(t) = \|F\|_2^2 < \infty. \end{aligned}$$

Thus $f \in L^2(\mathbf{F})$ and by definition,

$$(\mathscr{Z}f)(t, \cdot) = \sum_{k \in \mathbf{F}^\sharp} f(t+k)\, u_k = \sum_{k \in \mathbf{F}^\sharp} c_k(t)\, u_k = F(t, \cdot)$$

for each $t \in \tilde{I}$, where these series converge in $L^2(\mathbf{I})$ norm. Therefore, $\mathscr{Z}f = F$ and the proof is complete. ∎

There is a convolution which plays the same role for the Zak transform that the usual convolution plays for the Fourier transform. For $f \in L^1(\mathbf{F})$ and $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$ set

$$(f \odot g)(t+n) := \sum_{k \in \mathbf{F}^\#} f(t+k)\, g(t+n-k) \qquad (t \in \mathbf{I}, n \in \mathbf{F}^\#).$$

Since $\mathbf{F}^\#$ is a subgroup of $\mathbf{F}$, we have

$$\begin{aligned}\sum_{n \in \mathbf{F}^\#} \sum_{k \in \mathbf{F}^\#} \int_{\mathbf{I}} |f(t+k)\, g(t+n-k)|\, d\lambda(t) &= \sum_{n \in \mathbf{F}^\#} \sum_{k \in \mathbf{F}^\#} \int_{\mathbf{F}} |\chi \tau_k f|\, |\chi \tau_{n-k} g|\, d\lambda \\ &= \int_{\mathbf{F}} \left(\sum_{k \in \mathbf{F}^\#} |\chi \tau_k f| \right) \left(\sum_{l \in \mathbf{F}^\#} |\chi \tau_l g| \right) d\lambda.\end{aligned}$$

Hence by the definition of $\|\cdot\|_{\langle 1\infty \rangle}$, we see that $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$ implies $f \odot g \in L^1(\mathbf{F})$ with

$$\|f \odot g\|_1 \leqslant \|g\|_{\langle 1\infty \rangle} \int_{\mathbf{F}} \left(\sum_{k \in \mathbf{F}^\#} |\chi \tau_k f| \right) = \|g\|_{\langle 1\infty \rangle}\, \|f\|_1.$$

Therefore, for each $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$ the map $f \to f \odot g$ is a bounded linear operator from $L^1(\mathbf{F})$ into $L^1(\mathbf{F})$ whose operator norm is bounded by $\|g\|_{\langle 1\infty \rangle}$.

The same argument shows that

$$\sum_{n \in \mathbf{F}^\#} \sum_{k \in \mathbf{F}^\#} |f(t+k)\, g(t+n-k)| < \infty \tag{7}$$

for a.e. $[\lambda]$ $t \in \mathbf{I}$. This observation, in conjunction with (1), can be used to show that the Zak transform takes $\odot$ to the regular pointwise product.

THEOREM 2. *If $f \in L^1(\mathbf{F})$ and $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$, then $f \odot g \in L^1(\mathbf{F})$ and*

$$\mathscr{Z}(f \odot g) = \mathscr{Z}f\, \mathscr{Z}g. \tag{8}$$

Moreover, if g belongs to the smaller space $L^{\infty 1}(\mathbf{F})$, then the map $f \to g \odot f$ is a bounded linear operator from $L^p(\mathbf{F})$ into $L^p(\mathbf{F})$ for all $1 \leqslant p \leqslant \infty$. In fact, for such p's,

$$\|f \odot g\|_p \leqslant \|f\|_p\, \|g\|_{\infty 1}. \tag{9}$$

Proof. It is easy to check that the convolution $\odot$ is commutative, i.e., if $f \in L^1(\mathbf{F})$ and $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$, then

$$f \odot g = g \odot f.$$

Thus $\odot$ is a commutative bilinear operator and, by remarks above, the map $f \to g \odot f$ is a bounded linear operator from $L^1(\mathbf{F})$ into $L^1(\mathbf{F})$ for each fixed $g \in L^{\langle 1\infty \rangle}(\mathbf{F})$.

To verify (8), recall from (1) that $L^{\langle 1\infty \rangle}(\mathbf{F}) \subset L^1(\mathbf{F})$. Thus the three Zak transforms in (8) exist. Let $t \in \mathbf{I}$ be a point which satisfies (7) and $s \in \mathbf{I}$ be arbitrary. Since $\mathbf{F}^\#$ is a subgroup of $\mathbf{F}$, we have by definition that

$$\begin{aligned}
(\mathscr{Z}(f \odot g))(t, s) &= \sum_{n \in \mathbf{F}^\#} (f \odot g)(t+n)\, u_n(s) \\
&= \sum_{n \in \mathbf{F}^\#} \sum_{k \in \mathbf{F}^\#} f(t+k)\, g(t+n-k)\, u_n(s) \\
&= \sum_{k \in \mathbf{F}^\#} f(t+k)\, u_k(s) \sum_{n \in \mathbf{F}^\#} g(t+n-k)\, u_{n-k}(s) \\
&= (\mathscr{Z}f)(t, s)(\mathscr{Z}g)(t, s).
\end{aligned}$$

To verify (9), notice by commutativity and the generalized Minkowski inequality that

$$\begin{aligned}
\|g \odot f\|_p = \|f \odot g\|_p &= \left(\sum_{n \in \mathbf{F}^\#} \int_{\mathbf{I}} \left| \sum_{k \in \mathbf{F}^\#} \tau_k g \tau_{n-k} f \right|^p d\lambda \right)^{1/p} \\
&\leqslant \sum_{k \in \mathbf{F}^\#} \left(\int_{\mathbf{I}} \left(\sum_{n \in \mathbf{F}^\#} |\tau_k g \tau_{n-k} f|^p \right) d\lambda \right)^{1/p} \\
&= \sum_{k \in \mathbf{F}^\#} \left(\int_{\mathbf{I}} |\tau_k g|^p \sum_{l \in \mathbf{F}^\#} |\tau_l f|^p \, d\lambda \right)^{1/p} \\
&\leqslant \sum_{k \in \mathbf{F}^\#} \|\chi \tau_k g\|_\infty \left(\int_{\mathbf{I}} \sum_{l \in \mathbf{F}^\#} |\tau_l f|^p \, d\lambda \right)^{1/p} \\
&= \|g\|_{\infty 1} \, \|f\|_p. \quad \blacksquare
\end{aligned}$$

Theorem 2 also holds when $f \in L^2(\mathbf{F})$ and $g \in L^{\infty 1}(\mathbf{F})$. To see this, first observe by (2) that

$$\|\mathscr{Z}g\|_\infty \leqslant \|g\|_{\infty 1}$$

for each $g \in L^{\infty 1}(\mathbf{F})$ and, consequently, the maps

$$f \to \mathscr{Z}(f \odot g) \qquad \text{and} \qquad f \to (\mathscr{Z}f)(\mathscr{Z}g)$$

are bounded operators from $L^2(\mathbf{F})$ into $L^2(\mathbf{F})$. Since by (8), these operators coincide on the collection $L^1(\mathbf{F}) \cap L^2(\mathbf{F})$ and this last space is dense in $L^2(\mathbf{F})$, it follows that these operators coincide on $L^2(\mathbf{F})$, i.e., (8) holds for all $f \in L^2(\mathbf{F})$ and $g \in L^{\infty 1}(\mathbf{F})$.

4. WEIGHTED DILATIONS OF ADDITIVE CHARACTERS

For the case $\mathbf{F} = \mathbf{R}$, it has been useful (e.g., in the development of Weyl–Heisenberg frames) to consider systems of functions obtained by dilating additive characters and multiplying them by a weight function, i.e., systems of the form

$$\rho^b_{mn}(x) := u_m(b \cdot x)\, \rho(x-n) \qquad (x \in \mathbf{F},\, (m, n) \in \mathcal{N})$$

where $b \in \mathbf{F}$ and $\rho \in L^2(\mathbf{F})$.

When $b = 1$, the Zak transform can be used to decide whether the system $\rho_{m,n} := \rho^1_{m,n}$ is complete in both the classical and the 2-series case.

THEOREM 3. *The system* $(\rho_{mn}, (m, n) \in \mathcal{N})$ *is complete in* $L^2(\mathbf{F})$ *if and only if*

$$|\mathscr{Z}\rho| > 0 \qquad [\lambda] \text{ a.e.}$$

Proof. Because the transform $\mathscr{Z}$ is unitary and onto, it must map complete systems to complete systems. Let $u_m \otimes u_n$ represent the Kronecker product of the system u_n with itself, i.e., $(u_m \otimes u_n)(x, y) := u_m(x)\, u_n(y)$. Since the system $(u_n, n \in \mathbf{F}^{\#})$ is complete in $L^2(\mathbf{I})$, it follows that the system $(u_m \otimes u_n, (m, n) \in \mathcal{N})$ is complete in $L^2(\mathbf{I}^2)$. In view of (6), it suffices therefore to prove that

$$\langle f, \rho_{mn} \rangle = \langle \mathscr{Z}f\, \overline{\mathscr{Z}}\rho, u_m \otimes u_n \rangle \tag{10}$$

for each $m, n \in \mathbf{F}^{\#}$ and $f, \rho \in L^2(\mathbf{F})$.

We shall actually prove more. Since $\mathbf{F}^{\#}$ is a subgroup of $\mathbf{F}$ and $u_n(k) = 1$ for all $n, k \in \mathbf{F}^{\#}$, it is clear that

$$\begin{aligned}
(\mathscr{Z}\rho_{mn})(t, s) &= \sum_{k \in \mathbf{F}^{\#}} u_m(t+k)\, \rho(t+k-n)\, u_k(s) \\
&= u_m(t)\, u_n(s) \sum_{k \in \mathbf{F}^{\#}} \rho(t+k-n)\, u_{k-n}(s) \\
&= u_m(t)\, u_n(s) \sum_{l \in \mathbf{F}^{\#}} \rho(t+l)\, u_l(s) = u_m(t)\, u_n(s) (\mathscr{Z}\rho)(t, s)
\end{aligned}$$

for all $s, t \in \mathbf{I}$. Thus if $\rho \in L^1(\mathbf{F}) \cup L^2(\mathbf{F})$ then

$$\mathscr{Z}\rho_{mn} = u_m \otimes u_n \mathscr{Z}\rho \tag{11}$$

for all $(m, n) \in \mathscr{N}$.

It is now easy to verify (10). Indeed, since $\mathscr{Z}$ is unitary, (11) implies

$$\langle f, \rho_{mn} \rangle = \langle \mathscr{Z}f, \mathscr{Z}\rho_{mn} \rangle = \langle \mathscr{Z}f\, \overline{\mathscr{Z}}\rho, u_m \otimes u_n \rangle.$$

This proof illustrates a general principle: the map $\mathscr{Z}$ can be used to transform conditions on $L^2(\mathbf{F})$ to conditions on $L^2(\mathbf{I}^2)$. ∎

What happens to Theorem 3 when $b \neq 1$? For the case $\mathbf{F} = \mathbf{R}$, the system $(\rho^b_{mn}, (m, n) \in \mathscr{N})$ is never complete when $\|b\| > 1$. This is not an easy result and the proofs are different depending on whether b is rational or irrational. Daubechies [2] provided the proof for rational b but it does not work when b is irrational. A proof for this case can be based on a result of Rieffel [7] using von Neumann algebras.

For the case when $\mathbf{F}$ is the 2-series field, the following result holds.

THEOREM 4. *If $b \in \mathbf{F}^{\#}$ and $\|b\| > 1$, then the system ρ^b_{mn} is not complete for any choice of ρ.*

Proof. Let B represent the $L^2(\mathbf{I}^2)$ closure of the linear span of $\{u_{m \cdot b} \otimes u_n : (m, n) \in \mathscr{N}\}$. Since $\|b\| > 1$, B is a proper subspace of $L^2(\mathbf{I}^2)$. Moreover, it is easy to check that orthogonal projection from $L^2(\mathbf{I}^2)$ onto B is the conditional expectation operator $\mathscr{E}(\cdot \mid \mathscr{B}^b)$, where $\mathscr{B}^b$ is the σ-algebra generated by the function algebraic span of $\{u_{m \cdot b} \otimes u_n : (m, n) \in \mathscr{N}\}$. Since $b \in \mathbf{F}^{\#}$ implies $b \cdot m \in \mathbf{F}^{\#}$, it follows from (10) that

$$\begin{aligned} \langle f, \rho^b_{mn} \rangle &= \langle f, \rho_{m \cdot b, n} \rangle = \langle \mathscr{Z}f\, \overline{\mathscr{Z}}\rho, u_{m \cdot b} \otimes u_n \rangle \\ &= \langle \mathscr{E}(\mathscr{Z}f\, \overline{\mathscr{Z}}\rho \mid \mathscr{B}^b), u_{m \cdot b} \otimes u_n \rangle. \end{aligned}$$

Let F be a non-zero function in $L^2(\mathbf{I}^2)$ which is orthogonal to $\overline{\mathscr{Z}}\rho$ with respect to $\mathscr{B}^b$, i.e., such that

$$\mathscr{E}(F\overline{\mathscr{Z}}\rho \mid \mathscr{B}^b) = 0.$$

Since $\mathscr{Z}$ is 1–1 and onto, we can choose a non-zero $f \in L^2(\mathbf{F})$ such that $\mathscr{Z}f = F$. Therefore,

$$\langle f, \rho^b_{mn} \rangle = 0$$

for all $(m, n) \in \mathscr{N}$, i.e., the system ρ^b_{mn} is not complete. ∎

Whether the condition that b be an integer in Theorem 4 can be dropped in the 2-series case is an open question.

5. THE DUAL ZAK TRANSFORM

In this section, we introduce the dual Zak transform on both binary fields and its corresponding convolution. Let $\mathbf{F}$ represent the 2-adic or 2-series field and u be a basic character of $\mathbf{F}$. The *dual Zak transform* $\tilde{\mathscr{Z}}$ is defined analogously to the Zak transform, except the roles of $\mathbf{F}^{\#}$ and $\mathbf{I}$ are reversed. Namely, for each $f \in L^1(\mathbf{F})$, set

$$(\tilde{\mathscr{Z}} f)(k, l) := \int_{\mathbf{I}} \tau_k f \bar{u}_l \, d\lambda \qquad (k, l \in \mathbf{F}).$$

Since $\mathbf{I}$ is a subgroup of $\mathbf{F}$ we have

$$(\tilde{\mathscr{Z}} f)(k+t, l) = u_l(t)(\tilde{\mathscr{Z}} f)(k, l) \qquad (k, l \in \mathbf{F}^{\#}, t \in \mathbf{I}).$$

On the other hand, since $u = 1$ on $\mathbf{I}$ for both binary fields, we see that

$$(\tilde{\mathscr{Z}} f)(k, l+s) = (\tilde{\mathscr{Z}} f)(k, l) \qquad (k, l \in \mathbf{F}^{\#}, s \in \mathbf{I}).$$

Thus the values of $\tilde{\mathscr{Z}} f$ are uniquely determined by its values on the set $\mathbf{F}^{\#} \times \mathbf{F}^{\#}$.

Let $\chi_k := \chi_{I_0(k)}$, for $k \in \mathbf{F}^{\#}$, and observe that the collection $(\bar{u}_l(k)\, \chi_k u_l$, $(k, l) \in \mathbf{F}^{\#} \times \mathbf{F}^{\#})$ is a complete orthonormal system on $\mathbf{F}$. Since

$$(\tilde{\mathscr{Z}} f)(k, l) = \langle f, \bar{u}_l(k)\, \chi_k u_l \rangle \qquad ((k, l) \in \mathbf{F}^{\#} \times \mathbf{F}^{\#}),$$

the dual Zak transform of a function f can be interpreted as the Fourier coefficients of f with respect to this system. Therefore, if we let $L^p(\mathbf{F}^{\#} \times \mathbf{F}^{\#})$ denote the discrete L^p space of functions defined on $\mathbf{F}^{\#} \times \mathbf{F}^{\#}$, then the Riesz–Fischer theorem gives the following analogue of Theorem 1.

THEOREM 5. (i) *The map* $\tilde{\mathscr{Z}}$ *is a* 1–1 *bounded linear operator from* $L^1(\mathbf{F})$ *into* $L^{1\infty}(\mathbf{F}^{\#} \times \mathbf{F}^{\#})$ *and*

$$\|\tilde{\mathscr{Z}} f\|_{1\infty} \leqslant \|f\|_1 \qquad (f \in L^1(\mathbf{F})).$$

(ii) *The map* $\tilde{\mathscr{Z}}$ *is a unitary operator from* $L^2(\mathbf{F})$ *onto* $L^2(\mathbf{F}^{\#} \times \mathbf{F}^{\#})$ *and*

$$\langle \tilde{\mathscr{Z}} f, \tilde{\mathscr{Z}} g \rangle = \langle f, g \rangle \qquad (f, g \in L^1(\mathbf{F})),$$

where the left hand side represents the inner product on the Hilbert space $L^2(\mathbf{F}^{\#} \times \mathbf{F}^{\#})$.

Next, for $f, g \in L^1(\mathbf{F})$ define $f \tilde{\odot} g$ by

$$(f \tilde{\odot} g)(n+t) := \int_{\mathbf{I}} f(n+s)\, g(n+t-s)\, d\lambda(s) = (\chi\tau_n f * \tau_n g)(t)$$

for $n \in \mathbf{F}^\#$ and $t \in \mathbf{I}$. Since $\mathbf{I}$ is a subgroup of $\mathbf{F}$ and λ is translation invariant on $\mathbf{F}$, it is clear that

$$\begin{aligned}
&\sum_{n \in \mathbf{F}^\#} \int_{\mathbf{I}} \int_{\mathbf{I}} |f(n+s)\, g(n+t-s)|\, d\lambda(s)\, d\lambda(t) \\
&\quad = \int_{\mathbf{I}} \left(\sum_{n \in \mathbf{F}^\#} |f(n+s)| \int_{\mathbf{I}} |g(n+t-s)|\, d\lambda(t) \right) d\lambda(s) \\
&\quad \leqslant \sup_{n \in \mathbf{N}} \int_{\mathbf{F}} |\chi\tau_n g|\, d\lambda \left(\sum_{n \in \mathbf{F}^\#} |f(n+s)|\, d\lambda(s) \right) = \|g\|_{1\infty}\, \|f\|_1.
\end{aligned}$$

Therefore, $f \tilde{\odot} g$ is defined and belongs to $L^1(\mathbf{F})$ for every $f \in L^1(\mathbf{F})$ and $g \in L^{1\infty}(\mathbf{F})$, and for each fixed $g \in L^{1\infty}(\mathbf{F})$ the map $f \to f \tilde{\odot} g$ is a bounded linear operator from $L^1(\mathbf{F})$ into $L^1(\mathbf{F})$ with

$$\|f \tilde{\odot} g\|_1 \leqslant \|g\|_{1\infty}\, \|f\|_1.$$

A simple change of variables verifies that $\tilde{\odot}$ is a commutative binary operation. Thus for each fixed $g \in L^{1\infty}(\mathbf{F})$, the map $f \to g \tilde{\odot} f$ is a bounded linear operator from $L^1(\mathbf{F})$ into $L^1(\mathbf{F})$. As was the case for $\odot$, if we assume a stronger condition on g, this map is bounded on $L^p(\mathbf{F})$ for all $1 \leqslant p \leqslant \infty$. Indeed, let $g \in L^{\langle \infty 1 \rangle}$. Then by the generalized Minkowski inequality,

$$\begin{aligned}
\|f \tilde{\odot} g\|_p = \|g \tilde{\odot} f\|_p &= \left(\sum_{n \in \mathbf{F}^\#} \int_{\mathbf{I}} \left| \int_{\mathbf{I}} g(n+s)\, f(n+t-s)\, d\lambda(s) \right|^p d\lambda(t) \right)^{1/p} \\
&\leqslant \int_{\mathbf{I}} \left(\sum_{n \in \mathbf{F}^\#} \int_{\mathbf{I}} |g(n+s)\, f(n+t-s)|^p\, d\lambda(t) \right)^{1/p} d\lambda(s) \\
&\leqslant \int_{\mathbf{I}} \sup_{k \in \mathbf{F}^\#} |g(k+s)| \left(\sum_{n \in \mathbf{F}^\#} \int_{\mathbf{I}} |f(n+t-s)|^p\, d\lambda(t) \right)^{1/p} d\lambda(s) \\
&= \|f\|_p\, \|g\|_{\langle \infty 1 \rangle}.
\end{aligned}$$

Therefore,

$$\|f \tilde{\odot} g\|_p \leqslant \|g\|_{\langle \infty 1 \rangle}\, \|f\|_p.$$

The following result shows that the operation $\tilde{\odot}$ plays the same role for the dual Zak transform that convolution plays for the Fourier transform.

THEOREM 6. *If* $f \in L^1(\mathbf{F}) \cup L^2(\mathbf{F})$ *and* $g \in L^1(\mathbf{F})$ *then*

$$\tilde{\mathscr{Z}}(f \tilde{\odot} g) = \tilde{\mathscr{Z}} f \, \tilde{\mathscr{Z}} g.$$

Proof. Since $L^1(\mathbf{F}) \subset L^{\langle 1\infty \rangle}(\mathbf{F}) \subset L^{\infty 1}(\mathbf{F})$, the $\tilde{\mathscr{Z}}$ transforms of $f \tilde{\odot} g, f$, and g all exist. Fix $f \in L^1(\mathbf{F})$ and observe by definition, Fubini's theorem, and translation invariance of λ that

$$\begin{aligned}(\tilde{\mathscr{Z}}(f \tilde{\odot} g))(k, l) &= \int_{\mathbf{I}} \tau_k(f \tilde{\odot} g)\, \tau_k \bar{u}_l \, d\lambda \\ &= \int_{\mathbf{I}} \left(\int_{\mathbf{I}} f(k+s)\, g(k+t-s)\, d\lambda(s) \right) \bar{u}_l(t)\, d\lambda(t) \\ &= \int_{\mathbf{I}} f(k+s)\, \bar{u}_l(s) \left(\int_{\mathbf{I}} g(k+t-s)\, \bar{u}_l(t-s)\, d\lambda(t) \right) d\lambda(s) \\ &= (\tilde{\mathscr{Z}} f)(k, l)(\tilde{\mathscr{Z}} g)(k, l).\end{aligned}$$

for any $k, l \in \mathbf{F}^{\#}$. This proves Theorem 6 for the case $f \in L^1(\mathbf{F})$. Using the inequality

$$\|\tilde{\mathscr{Z}} g\|_{\infty} \leqslant \|g\|_{1\infty},$$

and repeating the argument which occupies the last paragraph in Section 3, we can establish Theorem 6 for the remaining case $f \in L^2(\mathbf{F})$. ∎

We close by noting that there is a close connection between the dual Zak transform of the functions

$$\tilde{\rho}^{\flat}_{mn}(x) := u_m(b \cdot x)\, \rho(x+n) \qquad (x \in \mathbf{F}, (m, n) \in \mathscr{N})$$

and $\tilde{\mathscr{Z}} \rho$. Indeed, for any $k, l \in \mathbf{F}^{\#}$,

$$\begin{aligned}(\tilde{\mathscr{Z}} \tilde{\rho}^{\flat}_{mn})(k, l) &= \int_{\mathbf{I}} \bar{u}_{m \cdot b}(x+k)\, \rho(x+n+k)\, \bar{u}_l(x)\, d\lambda(x) \\ &= \bar{u}_m(b \cdot k) \int_{\mathbf{I}} \rho(x+n+k)\, \bar{u}_{l+m \cdot b}(x)\, d\lambda(x) \\ &= \bar{u}_m(b \cdot k)(\tilde{\mathscr{Z}} \rho)(n+k, l+m \cdot b).\end{aligned}$$

Thus there is a connection between the Zak transform and "dyadic wavelets" defined on binary fields.

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